Written by Yarden Nativ, mail: nativyardenac@gmail.com

11th Recitation 08.06.23

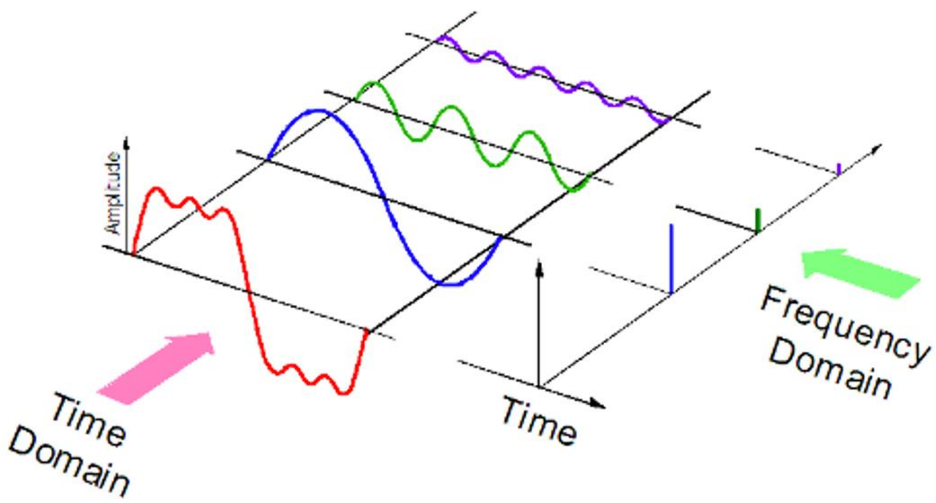
Frequency Domain I: Fourier transform, Nyquist Theorem

Fourier Transform

Fourier transform is the basic tool in spectral analysis to allow a transformation between the time domain and frequency domain. As a transformation, it allows us to see the data in two dimensions, in which one of them indicates what frequencies the signal is made off. Those frequencies can be represented either by delta functions or with interpolation between the dots. Sometimes we use Fourier transform to compute more easily some operations like Convolution.

In brain research we commonly use the Fourier transform along a wide range of fields- betta oscillations in Parkinson disease, Gamma oscillations in attention studies and so on.

Some illustration of the Fourier transform idea:



Very cool simulation- http://www.falstad.com/fourier/

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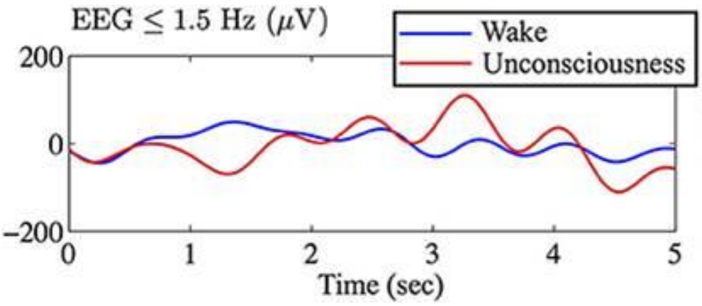
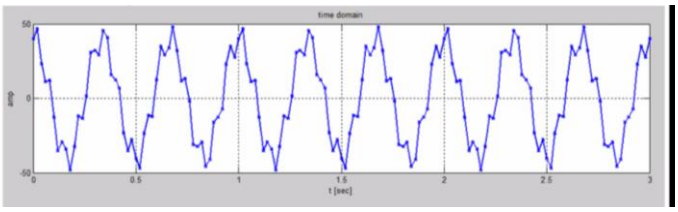
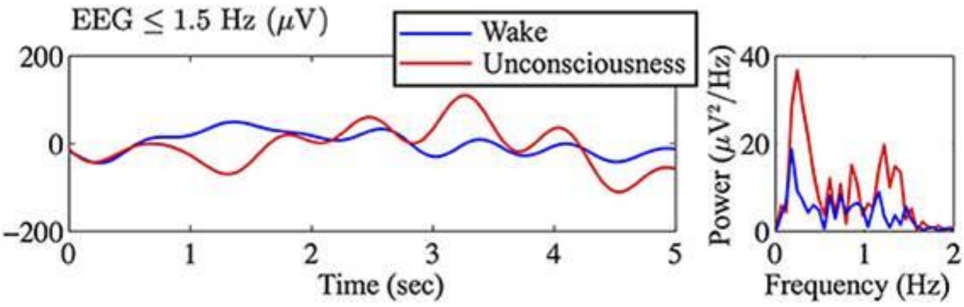
Class exercise:

Given the following signals, what frequencies do you notice at it?

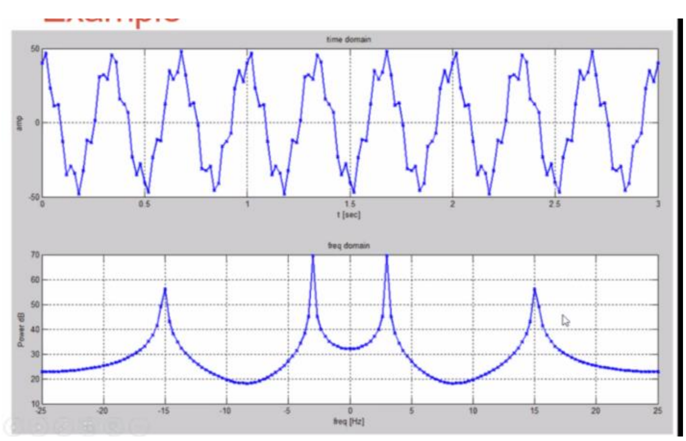




Solution:



(adopted from: https://biology.stackexchange.com/questions/44955/why-is-fast-fourier-transform-applied-to-raw-eeg-data)



Definitions

Euler's formula:

$$e^{ix} = \cos(x) + isin(x)$$

$$\sin(x) = Im\left(e^{ix}\right) = \frac{e^{ix} - e^{-ix}}{2i} \qquad cos(x) = Re\left(e^{ix}\right) = \frac{e^{ix} + e^{-ix}}{2}$$

Note: Sine and cosine functions.

Note: Why do we prefer the complex representation? Because absolute value of the transform represents magnitude/amplitude, and arctangent the phase (in polar coordinates).

Note: Reminder of decibels, it is a scale to show differences in magnitudes, based on the calculation $20\log_{10}\left(\frac{Magnitude_1}{Magnitude_2}\right)$ so that if $Magnitude_1 = 10Magnitude_2$, than $db = 20$.

Continuous transform:

$$\tilde{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{X}(\omega) \cdot e^{i\omega t}d\omega$$

Note: can be ran in a moving window to show changes in the frequencies

Discrete transform (practically in use):

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-\frac{2\pi kn}{N}i}$$

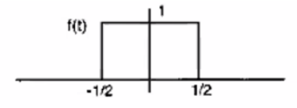
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] \cdot e^{-\frac{2\pi kn}{N}i}$$

Note: the number of samples in the frequency domain equals to the number of samples in the time domain.

Class exercise:

Calculate the Fourier transform of the step function $f = rect(t)$ when the step values are 1 and the step width is between $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Illustration:

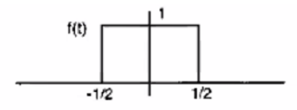


Solution:

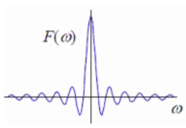
$$F(\omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\omega t} dt = \frac{1}{-i\omega}\left[e^{-i\omega t}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = -\frac{1}{i\omega}\left[e^{-\frac{i\omega}{2}} - e^{\frac{i\omega}{2}}\right] = \frac{1}{\left(\frac{\omega}{2}\right)} \frac{e^{\frac{i\omega}{2}} - e^{\frac{i\omega}{2}}}{2i} = \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}$$

$$\equiv sinc\left(\frac{\omega}{2}\right)$$

Visually:



transforms to



Some useful calculations:

TABLE A.2 Fourier-Transform Pairs

Time Function	Fourier Transform
$\text{rect}\left(\frac{t}{T}\right)$	$T\text{sinc}(fT)$
$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases} 1-\frac{\lvert t\rvert}{T}, & \lvert t\rvert < T \\ 0, & \lvert t\rvert \geq T \end{cases}$	$T\text{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$g(t-t_0)$	$\exp(j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$

Notes: $\delta(t)$ = delta function, or unit impulse
$\text{rect}(t)$ = rectangular function of unit amplitude and unit duration centered on the origin
$\text{sinc}(t)$ = sinc function

Nyquist sampling theorem

As we've seen in all the examples above, the sampling frequency can affect the results of the transform. For example, the time of the recitation lesson, Thursdays at 12:00, can be referred as a periodic process. If someone would visit the room every Thursday at 12:30 for one minute, he might conclude that there is always a recitation class in the room at any other given time. To the luck of the students, this is not the true. If this visitor would sample even once more during the week, for examples at Mondays at 12:30, he would be able to track concisely the periodic process.

In short, the **Nyquist Theorem** says that you need at least 2 samples per "cycle" of your input signal to define it. You can accurately measure the frequency of a signal with frequency f as long as you are sampling it at greater than $2f$. If you try to measure the frequency of signals having a frequency above f with a sampler operating at $2f$, you will **alias** the signal, or create false images of this signal at frequencies below f. These false frequencies will appear as mirror images of the original frequency around the Nyquist frequency. This situation is called "aliasing back" or "folding back".

Illustration:

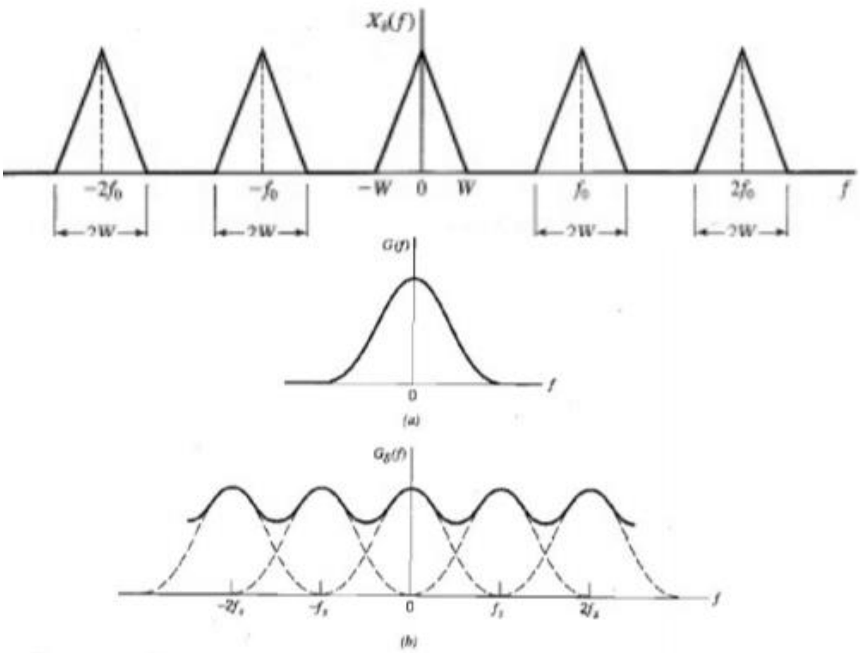


FIGURE 3.3 (a) Spectrum of a signal. (b) Spectrum of an undersampled version of the signal exhibiting the aliasing phenomenon.

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Class exercise: example from exam 2006

The electrical potential generated by the Electrical Frog may be described by the function $V(t) = 1 + Xsin(50t \cdot 2\pi) + Ycos(70t \cdot 2\pi)$.

a. Assuming that the scientist samples the potential at 120 Hz, draw the spectrum of the sampled signal – V(ω).
b. Assuming that the sampling rate cannot increase. Provide a solution for extracting X and Y.

Solution:

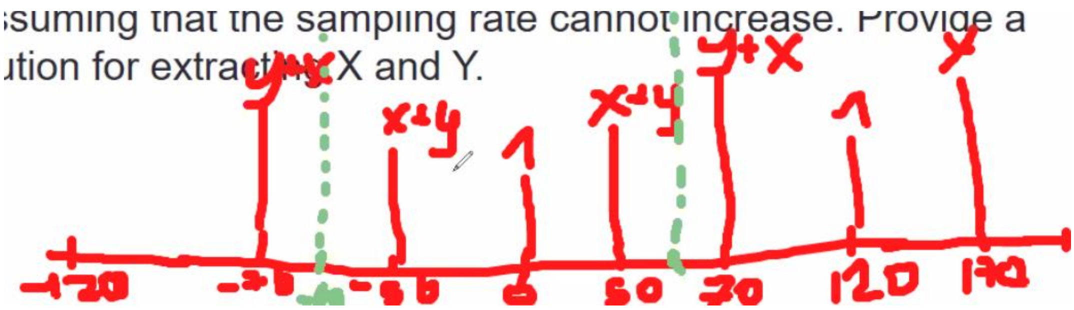
a. Given the signal, we expect the transformation to show to frequencies, 50Hz and 70Hz with magnitudes X and Y correspondingly. Therefore, according to Nyquist theorem, the minimal sampling rate we need to use is 100Hz for the 50Hz and 140Hz for the 70Hz. But we have only 120Hz sampling, and therefore we will have an aliasing. How to calculate it?

1. The aliasing will be symmetrical to $50 + 70 = 120$.

2. There will be unreal magnitudes in the frequencies of 120 ± 50, so at 70 and 170, and unreal magnitudes in the frequencies of 120 ± 70, so at 50 and 190.

3. Due to the sampling rate of 120Hz we will see the result of the Fourier transform only in the range of $[-60: 60]$.

Therefore the only difference that we will ses is only one frequency with a magnitude very higher than X.



b. Given the right sampling rate, we can use the aliasing for our purposes. For example in a sampling rate of 90 we will have aliasing of the frequencies to 20 and 40, so that those will be false frequencies but with the right magnitudes of X and Y. We can repeat this procedure, and by so calculate the frequencies based on the different results we get.

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Class exercise: example from exam 2006

Neuron X fires at a mean rate of 2 spikes/s and its spectrum has a peak only around 9Hz and neuron Y fires at a mean rate of 9 spikes/s and its spectrum has a peak only around 2Hz.

a. X & Y are possible

b. X & Y are impossible

c. X is possible & Y is not.

d. Y is possible & X is not.

Find fs (sampling frequency) for the possible scenario

Solution:

For neuron X the minimal sampling rate is 18 Hz and for neuron Y 4 Hz, therefore the minimal sampling rate is 18 Hz for both. In such a case, we should have had another peak for neuron X at 2Hz, but we don't have such. Therefore, X can't be true. In contrast, for Y if we have sampling frequency of 11 Hz or 7 Hz, we have aliasing at $\pm 2Hz$ (11-9) and then we can see it in the range of $\pm 5Hz$.